Maxima and Minima

Differentiation can be used to find the maximum and minimum values of a function. Because the derivative provides information about the slope a function we can use it to locate points on a graph where the slope is zero. Such points are often associated with the largest or smallest values of the function. In many applications, an economist will be interested in such points for obvious reasons such as maximising profit, or minimising losses or costs, for example.

Stationary Points

Often, we are interested in the ups and downs of a function, its maximum and minimum values, its turning points. Drawing a graph of a function will reveal this behaviour, but if we want to know the precise location of such points we need to turn to use differential calculus.

Consider the graph of the function, $y(x)$, shown in Figure 1. If, at the points marked A, B and C, we draw tangents to the graph, note that these are parallel to the x axis. They are horizontal. This means that at each of the points A, B and C the gradient of the graph is zero. Consider the graph of the function, $y(x)$, shown in Figure 1. If, at the points marked A, B and C, we draw tangents to the graph, note that these are parallel to the x axis. They are horizontal. This means that at each of the points A, B and C the gradient of the graph is zero.

![Graph showing stationary points A, B, and C](image)

**Figure 1: The slope of this graph is zero at each of the points A, B and C**
The slope of a graph is given by \( \frac{dy}{dx} \). Consequently, \( \frac{dy}{dx} = 0 \) at points A, B and C. All of these points are known as stationary points.

Any point at which the tangent to the graph is horizontal is called a stationary point. We can locate stationary points by looking for points at which \( \frac{dy}{dx} = 0 \).

**Turning Points**

Refer again to Figure 1. Notice that at points A and B the curve actually turns. These two stationary points are referred to as turning points. Point C is not a turning point because, although the graph is flat for a short time, the curve continues to go down as we look from left to right.

So, all turning points are stationary points. But not all stationary points are turning points (e.g. point C).

In other words, there are points for which \( \frac{dy}{dx} = 0 \) which are not turning points.

At a turning point \( \frac{dy}{dx} = 0 \). However, not all points where \( \frac{dy}{dx} = 0 \) are turning points, i.e. not all stationary points are turning points.

Point A in Figure 1 is called a local maximum because in its immediate area it is the highest point, and so represents the greatest or maximum value of the function. Point B in Figure 1 is called a local minimum because in its immediate area it is the lowest point, and so represents the least, or minimum, value of the function. Loosely speaking, we refer to a local maximum as simply a maximum. Similarly, a local minimum is often just called a minimum.

**Distinguishing Maximum Points from Minimum Points**

Think about what happens to the gradient of the graph as we travel through the minimum turning point, from left to right, that is as \( x \) increases. Study Figure 2 to help you do this.
Notice that to the left of the minimum point, \( \frac{dy}{dx} \) is negative because the tangent has a negative slope.

At the minimum point, \( \frac{dy}{dx} = 0 \).

To the right of the minimum point \( \frac{dy}{dx} \) is positive, because here the tangent has a positive slope.

So, \( \frac{dy}{dx} \) goes from negative, to zero, to positive as \( x \) increases.

In other words, \( \frac{dy}{dx} \) must be increasing as \( x \) increases.

We can use this observation, once we have found a stationary point, to check if the point is a minimum.

If \( \frac{dy}{dx} \) is increasing near the stationary point then that point must be minimum.

If the derivative of \( \frac{dy}{dx} \) is positive then we will know that \( \frac{dy}{dx} \) is increasing; so we will know that the stationary point is a minimum.

The derivative of \( \frac{dy}{dx} \), called the second derivative, is written \( \frac{d^{2}y}{dx^{2}} \).

We conclude that if \( \frac{d^{2}y}{dx^{2}} \) is positive at a stationary point, then that point must be a minimum turning point.

The opposite applies to a maximum turning point.

**The Second Derivative Test: Summary**

1. Locate the position of stationary points by looking for points where \( \frac{dy}{dx} = 0 \).
2. Calculate \( \frac{d^{2}y}{dx^{2}} \) at each point we find.
3. If \( \frac{d^2y}{dx^2} \) is positive then the stationary point is a minimum turning point.

4. If \( \frac{d^2y}{dx^2} \) is negative, then the point is a maximum turning point.

Example

Suppose we wish to find the turning points of the function \( y = x^3 - 3x + 2 \) and distinguish between them.

We need to find where the turning points are, and whether we have maximum or minimum points.

First of all we carry out the differentiation and set \( \frac{dy}{dx} \) equal to zero. This will enable us to look for any stationary points, including any turning points.

\[
y = x^3 - 3x + 2
\]
\[
\frac{dy}{dx} = 3x^2 - 3
\]

At stationary points, \( \frac{dy}{dx} = 0 \) and so

\[
3x^2 - 3 = 0
\]
\[
(3x - 3)(x + 1) = 0
\]
\[
3x - 3 = 0 \quad x + 1 = 0
\]
\[
x = 1 \quad x = -1
\]

When \( x = 1 \) \quad \( y = 1^3 - 3(1) + 2 = 0 \)

When \( x = -1 \) \quad \( y = (-1)^3 - 3(-1) + 2 = 4 \)

Two stationary points occur at \((1, 0)\) and \((-1, 4)\).

Next we need to determine whether we have maximum or minimum points, or possibly points such as \( C \) in Figure 1 which are neither maxima nor minima.

We have seen that the first derivative is

\[
\frac{dy}{dx} = 3x^2 - 3
\]

Differentiating this we can find the second derivative

\[
\frac{d^2y}{dx^2} = 6x
\]

We now take each point in turn and use our test.

When \( x = 1 \)
We are not really interested in this value. What is important is its sign. Because it is positive we know we are dealing with a minimum point.

When \( x = -1 \)

\[
\frac{d^2y}{dx^2} = 6
\]

Again, what is important is its sign. Because it is negative we have a maximum point.

Finally, to finish this off we can produce a quick sketch of the function now that we know the precise locations of its two turning points.

Figure 3: Graph of \( y = x^3 + 3x - 2 \) showing the turning points

Note: If \( d^2y/dx^2 = 0 \) it is possible that we have a maximum, or a minimum, or indeed other sorts of behaviour. So if \( d^2y/dx^2 = 0 \) the second derivative test does not give us useful information and we must use an alternative method, using the first derivative.
Using the First Derivative to Distinguish Maxima and Minima

Example

Suppose we wish to find the turning points of the function below and distinguish between them.

\[ y = \frac{(x - 1)^2}{x} \]

First of all we need to find \( \frac{dy}{dx} \).

In this case we need to apply the quotient rule for differentiation.

\[
\frac{dy}{dx} = \frac{x \cdot 2(x - 1) - (x - 1)^2 \cdot 1}{x^2} = \frac{x(2x - 2) - (x - 1)^2}{x^2}
\]

\[
\frac{dy}{dx} = \frac{(2x^2 - 2x) - (x - 1)(x - 1)}{x^2}
\]

\[
\frac{dy}{dx} = \frac{(2x^2 - 2x) - (x^2 - 2x + 1)}{x^2}
\]

\[
\frac{dy}{dx} = \frac{2x^2 - 2x - x^2 + 2x - 1}{x^2}
\]

\[
\frac{dy}{dx} = \frac{x^2 - 1}{x^2}
\]

\[
\frac{dy}{dx} = \frac{(x - 1)(x + 1)}{x^2}
\]

We now set \( \frac{dy}{dx} \) equal to zero in order to locate the stationary points including any turning points.

When equating a fraction to zero, it is the top line, the numerator, which must equal zero. Therefore

\[(x - 1)(x + 1) = 0\]

\[x = 1 \quad x = -1\]

When \( x = 1 \)

\[y = \frac{(1 - 1)^2}{1} = 0\]

When \( x = -1 \)

\[y = \frac{(-1 - 1)^2}{-1} = -4\]

Stationary points occur at \((1, 0)\) and \((-1, -4)\).
We now have to decide whether these are maximum points or minimum points. We could calculate $d^2y/dx^2$ and use the second derivative test as in the previous example. This would involve differentiating $(x - 1)(x + 1)/x^2$ which is possible but not necessary as there is an alternative way.

We can look at how $dy/dx$ changes as we move through the stationary point. In essence, we can find out what happens to $d^2y/dx^2$ without actually calculating it.

First consider the point at $x = -1$.

We look at what is happening a little bit before the point where $x = -1$, and a little bit afterwards.

We express the idea of ‘a little bit before’ and ‘a little bit afterwards’ in the following way. We can write $-1 - \varepsilon$ to represent a little bit less than $-1$, and $-1 + \varepsilon$ to represent a little bit more. The symbol $\varepsilon$ is the Greek letter epsilon. It represents a small positive quantity, say 0.1.

Then $-1 - \varepsilon$ would be $-1.1$, just a little less than $-1$. Similarly $-1 + \varepsilon$ would be $-0.9$, just a little more than $-1$.

We now have a look at $dy/dx$; not its value, but its sign.

When $x = -1 - \varepsilon$, say $-1.1$, $dy/dx$ is positive

\[
\frac{dy}{dx} = \frac{(x - 1)(x + 1)}{x^2}
\]

\[
\frac{dy}{dx} = \frac{(-1.1 - 1)(-1.1 + 1)}{-1.1^2}
\]

\[
\frac{dy}{dx} = \frac{(-2.1)(-0.1)}{1.21} = +0.17
\]

When $x = -1 + \varepsilon$, say $-0.9$, $dy/dx$ is negative

\[
\frac{dy}{dx} = \frac{(x - 1)(x + 1)}{x^2}
\]

\[
\frac{dy}{dx} = \frac{(-0.9 - 1)(-0.9 + 1)}{-0.9^2}
\]

\[
\frac{dy}{dx} = \frac{(-1.9)(0.1)}{0.81} = -0.23
\]

When $x = -1$ we already know that $dy/dx = 0$

We can summarise this information in Table 1.
Table 1 shows us that the stationary point at \((-1, -4)\) is a maximum turning point.

Then we turn to the point \((1, 0)\). We carry out a similar analysis, looking at the sign of \(\frac{dy}{dx}\) at \(x = 1 - \varepsilon\), \(x = 1\), and \(x = 1 + \varepsilon\). The results are summarised in Table 2.

<table>
<thead>
<tr>
<th>(x = 1 - \varepsilon)</th>
<th>(x = 1)</th>
<th>(x = 1 + \varepsilon)</th>
</tr>
</thead>
<tbody>
<tr>
<td>sign of (\frac{dy}{dx})</td>
<td>-</td>
<td>0</td>
</tr>
<tr>
<td>shape of graph</td>
<td>(\searrow)</td>
<td>(\rightarrow)</td>
</tr>
</tbody>
</table>

Table 2: Behaviour of the graph near the point \((1, 0)\)

We see that the point is a minimum.

This, so-called first derivative test, is also the way to do it if \(d^2y/dx^2\) is zero in which case the second derivative test does not work.

Finally, for completeness a graph of \(y = (x - 1)^2/x\) is shown in Figure 4 where you can see the maximum and minimum points.
Figure 4: A graph of $y = \frac{(x - 1)^2}{x}$ showing the turning points.